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SUMMARY

Unsteady lift functions for wings of finite aspect ratio have been calculated by approximate methods involving corrections of the aerodynamic inertia and of the angle of the infinite wing.

The starting lift of the finite wing is found to be only slightly less than that of the infinite wing; whereas the final lift may be considerably less. The calculations indicate that the distribution of lift near the start is similar to the final distribution.

Both the indicial and the oscillating lift functions are given. Approximate operational equivalents of the functions have been devised to facilitate the calculation of lift under various conditions of motion.

INTRODUCTION

The two-dimensional potential theory of airfoils in unsteady motion was set forth by Wagner (reference 1) and has been extended to problems involving the motion of hinged or flexible airfoils by Theodorsen (reference 2) and Küssner (reference 3).

It is known that, in the case of steady motion, a correction is necessary before the results of the two-dimensional theory can be applied to actual wings of finite aspect ratio. No corresponding correction has, however, been developed for the conditions of nonuniform motion.* The calculations contained herein lead to such a correction, which is applicable, within the usual limitations, to arbitrary dynamical conditions of motion.

*A discussion in general terms of such a correction has been given in a recent paper by Cicala (reference 4).

THE INDICIAL LIFT

Influence of the Wake

Owing to the presence of circulation, the lifting wing leaves in its wake a surface of discontinuity whose local vortex strength will be determined by the rate of change of circulation taken both across the span and along the flight path. (See fig. 1.) The formation of such a wake and the configuration of the vortices in the wake are determined by the assumption that the flow at each instant conforms to the Kutta condition. An essential feature of the problem is the determination of the influence of this wake on the wing.

It is important to note that the wake is supposed to remain plane and undistorted. The consequence of this assumption is that the effects of the wake are additive, permitting the various flows to be built up by superposition.

Thus, if the solution is known for the case of a sudden start of the motion - or what amounts to the same thing under the assumptions involved, a sudden change in angle of attack - this solution may be used as the element in an integral that gives the lift in any variable motion. With this point in mind, attention will first be directed to the special case in which the wing starts suddenly from rest at $t = 0$ and maintains the uniform flight velocity V with the normal velocity w thereafter.

Before the three-dimensional problem is considered, it will be helpful to review certain aspects of the two-dimensional theory. Figure 2 shows the elemental flow used as a starting point by Wagner (reference 1).

This flow is caused by two vortices, representing an element of circulation around the wing and the vortex left in the wake when this circulation was generated. The streamlines of this flow are eccentric circles and one such circle of unit radius is chosen to represent the wing section. The axes are then placed so that this circle has its center at the origin. The geometry of the pattern is such that, when the wake vortex is at z, the wing vortex must be at $1/z$. Such spacing preserves the unit circle as a streamline of the flow.

Transformation of the pattern by the formula

$$2\zeta = z + \frac{1}{z} \tag{1}$$

flattens the unit circle into a thin-line wing section and distorts the originally circular streamlines into oval Joukowski figures (fig. 2).

The transformed pattern thus represents the circulatory flow about a series of symmetrical Joukowski airfoils with an associated countervortex in the wake. For convenience, the ideally thin airfoil is chosen.

Each elementary flow of the type shown (fig. 2) contributes a certain velocity around the trailing edge of the airfoil. An instantaneous flow due to an angle of attack of the airfoil also gives a trailing-edge velocity, but of opposite sense. On this basis, the problem of circulation with varying angle of attack may be solved by an inverse procedure. Assume some convenient distribution of wake vorticity and calculate the trailing-edge velocity at each point along the flight path corresponding to the formation of the prescribed wake. The particular variation of angle of attack necessary to cancel this trailing-edge flow at each instant (Kutta condition) can then be determined. If a number of such curves are found, they may be added in various proportions so as to approximate any prescribed variation of angle of attack; the corresponding curves of variation of circulation along the flight path are then added in like proportions.

It was in essentially the manner just described that Wagner (reference 1) built up the flow produced by a wing during the uniform motion following a sudden unit change in angle of attack. The integrated pressures over the airfoil give a lift coefficient that approaches asymptotically the known steady value 2π. The starting lift is found to be exactly one-half this value. The center of pressure remains at the quarter-chord point throughout the motion.

In the case of the finite wing, an element of the wake will be as depicted in figure 2 but will, in addition, contain vortices completing each circuit to the wing through the tips. Near the start of the motion, these tip vortices will be short and their influence on the wing will

consequently be small. Hence, the starting lift of the finite wing will be very nearly equal to that of the infinite wing. As the wake lengthens, the correction will increase and finally approach the magnitude given by the Prandtl theory.

In the present problem, it is desired to follow along the lines indicated by the Prandtl theory insofar as possible, using the existing two-dimensional theory as a basis and determining the effect of aspect ratio as a correction. With long tip vortices and with smooth distributions of load, calculations show nearly uniform distributions of downwash over the wing. With a short wake, however, the calculations show proportionately much greater curvature of the induced flow so that the effects near the start cannot be approximated by a simple angle-of-attack correction as in the Prandtl theory.

Lift Near the Start

The starting lift of any wing may be expressed in a very simple manner based on the Kutta condition. It is seen that, as a consequence of this condition, the portion of wake adjacent to the trailing edge must move as an impermeable extension of the wing surface. The flow produced at the first instant is the same as that caused by the wing in process of growing wider at the rate V. It follows that the starting lift may be thought of as the reaction to uniform motion of a wing with increasing mass:

$$L = w \frac{dm'}{dt} \tag{2}$$

where m' is the mass representing the aerodynamic inertia of the wing.

In order to apply equation (2) to the finite wing, it is necessary to know the inertia factor for such a wing as a function of the width. In general, it would therefore be necessary to determine the potential flow for normal motion of each wing of given shape. Solutions are provided, however, by classical hydrodynamic theory for elliptic plates, and it is possible to use these solutions to represent approximately the initial rate of increase of inertia of a wing of conventional shape.

The distribution of potential over each chordwise section of an elliptic plate in normal motion has the same form as the corresponding two-dimensional potential. At a certain value of the normal velocity, this distribution is given by a circle with the chord of the section as the diameter. Thus

$$\varphi = w\sqrt{1 - \zeta^2}$$

In two-dimensional flow, the normal velocity is 1.00 and it is slightly greater for the finite disk. The factor of increase is the ratio of the semiperimeter to the span, given by the elliptic integral E, i.e.,

$$\varphi = \frac{w}{E}\sqrt{1 - \zeta^2}$$

If the edge of the plate distorts into a slightly wider ellipse (fig. 3), the potential differences arising will be measured by the differences between the ordinates of the original and of the slightly expanded circles. The change in the factor E may be neglected. The pressure differential at any point is given by the formula

$$p = -2\rho\left[V\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial t}\right] \tag{3}$$

where

$$\left.\begin{aligned} \varphi &= \sqrt{1 - x^2} = \sin\theta \\ \frac{\partial\varphi}{\partial x} &= \frac{-x}{\sqrt{1 - x^2}} = -\cot\theta \end{aligned}\right\} \tag{4}$$

and, from the geometry of the circle,

$$\left.\frac{\partial\varphi}{\partial t}\right|_{t=o} = \left.\frac{\partial\varphi}{\partial\Delta c}\frac{d\Delta c}{dt}\right|_{t=o} = V\left.\frac{\partial\varphi}{\partial\Delta c}\right|_{\Delta c=o} = V\frac{1}{2}\cot\frac{1}{2}\theta \tag{5}$$

The pressure across the plate with normal velocity $w = E$ and flight velocity V is, therefore,

$$p_{t=o} = \rho V \left[2 \cot \theta - \cot \frac{\theta}{2} \right] \tag{6}$$

Integration of this pressure over any section gives the lift coefficient

$$C_{L_{t=o}} = \frac{\pi}{E} \frac{w}{V} = \frac{\pi}{E} \alpha \tag{7}$$

with each local center of pressure at the quarter-chord line.

The start of the elliptic wing is equivalent to a uniform lengthening of each chord, so that the true elliptic outline is not preserved. It may be shown, however, that such a change does conform very nearly to a change into another slightly larger ellipse at all points except those near the tips. Furthermore, if it is assumed that the wing is distorted in any of a number of ways into a slightly different elliptical plan form, it is found that the change of virtual inertia is but little affected by the change in shape and depends primarily on the over-all change in size. Each such distortion can be thought of as representing a certain distribution of the starting velocity V around the edge of the wing. Equation (7) is exact for all these distributions and, since they may be made to fall closely on either side of the curve V = constant (representing the start of a rigid wing), the equation is considered applicable to this case also.

The Downwash Correction

A fairly accurate curve of the growth of lift might be drawn by connecting the starting value given by equation (7) asymptotically to the known steady value. After the wing has progressed several chord lengths, however, the effect of the vortex wake can be treated simply as a modification of the angle of attack and in this way it will be possible to obtain a closer approach to the true form of the curve.

Figure 4 shows how an elementary loop vortex in the

wake of the finite wing can be formed by cancelation from an element of the wake of the infinite wing. The flow produced by segments BCD GFH accounts for the aspect-ratio effect.

The relation determining the spacing of the vortices A and E (fig. 4) is found in the two-dimensional theory. (See fig. 2.) At the instant an element of circulation is acquired by the wing, the bound vortex and the counter wake vortex coincide at the point corresponding to the trailing edge of the airfoil. The wake vortex is subsequently carried downstream with the fluid at the velocity V while the bound vortex moves ahead into the wing as shown. It is important to note that the position of the centroid of the wing circulation is unchanged by the transformation from the circular to the flat wing sections. Thus, when the wake vortex is at ζ, the flattened bound vortex retains its centroid at $1/z$. The equivalent length x of the tip vortices, in terms of the flight-path distance s, becomes

$$x = \frac{1}{2}\left[1 + s + \sqrt{s^2 + 2s} - \frac{1}{1 + s + \sqrt{s^2 + 2s}}\right] = \sqrt{s(s + 2)} \qquad (8)$$

The tip vortices do not, of course, terminate abruptly but merge into the wing. The effective length x gives a good approximation to the effect at some distance. Figure 5 illustrates the rapid spread of the discontinuity into the wing subsequent to its origin at the trailing edge.

Since the vortex wake is supposed to remain plane and undistorted, the principle of superposition can be applied to the calculation of downwash $w_i(s)$. Then the downwash due to any distribution of vorticity along the flight path and over the span can be computed by integration of the effects of elementary lines of the pattern indicated in figure 4. In the simplest aspect of the problem, the surface of discontinuity may be replaced by a "skeleton" composed of two tip vortices of finite strength connected across the span by a vortex sheet. The tip vortices are located in such a way that their influence at the middle of the wing approximates the influence of the spanwise grading of discontinuity. The problem will first be treated in this manner and will later be extended to include the elliptical form of spanwise loading.

A straight downwash flow at the center being assumed, the calculation for an elementary loop is carried out at the chordwise centroid of the wing vorticity where the small influence of segments B and G disappears. By the application of Biot-Savart's rule, the downwash due to segments DC and FH is found to be

$$\frac{dw_i}{d\gamma}(s) = \frac{1}{2\pi}\left[\left(\frac{x}{y} + \frac{y}{x}\right)\frac{1}{\sqrt{x^2 + y^2}} - \frac{1}{x}\right] \tag{9}$$

The transformation of the length x to the distance s is accomplished by equation (8).

With the tip vortices at the fixed spacing y, the downwash due to any variation of vorticity along the flight path $\gamma(s)$ may be found from

$$w_i(s) = \frac{dw_i}{d\gamma}(s)\ \gamma(0) + \int_0^s \frac{dw_i}{d\gamma}(s - s_o)\ \gamma'(s_o)\ ds_o \tag{10}$$

Circulation and Lift

The circulation around the wing at later stages of the motion may be determined from the two-dimensional theory by using the effective angle of attack

$$\alpha_e = \alpha - \alpha_i \tag{11}$$

where

$$\alpha_i = \frac{w_i}{V}$$

Let $\Gamma_1(s)$ be the rise of circulation following a unit jump of angle of attack (indicial circulation) as given by Wagner (reference 1) for the infinite wing. (The subscript $_1$ is used hereinafter to denote the effect of a unit jump of angle of attack occurring at $s = 0$.) Then, for the finite wing,

$$\gamma(s) = \Gamma_1(s)\ \alpha_e(0) + \int_0^s \Gamma_1(s - s_o)\ \alpha_e'(s_o)\ ds_o \tag{12}$$

The determination of the effective angle of attack and the circulation of the finite wing thus depend on the simultaneous solution of integral equations (10) and (12).

The formal solution of equations (10), (11), and (12) for the indicial downwash $w_{i_1}(s)$ and for the indicial circulation $\gamma_1(s)$ would be expected to be quite difficult. It is, however, a fairly easy matter to find satisfactory curves by trial, particularly since it is known that $\gamma_1(s)$ coincides with $\Gamma_1(s)$ and that $w_{i_1}(s) = 0$ at the beginning $(s = 0)$. Figures 6 and 7 show curves determined in this way for two different values of y, corresponding to aspect ratios 3 and 6 $\left(y = \frac{2}{\pi}\frac{b}{2}\right.$ for the elliptical wing).

The lift at the later stages of the motion is found by combining the effective angle of attack $\alpha_e(s)$ with the two-dimensional indicial-lift function $c_{l_1}(s)$ of Wagner. The total indicial lift for the three-dimensional wing, neglecting the small increment due to acceleration of the downwash flow, is therefore

$$C_{L_1}(s) = c_{l_1}(s) + \int_0^s c_{l_1}(s - s_0)\,\alpha_{e_1}'(s_0)ds_0 \qquad (13)$$

Plots of C_{L_1} determined by graphical integration are given in figure 8.

The lift of the finite wing approaches a steady value much more rapidly than indicated by Wagner's curve. It is observed that the downwash due to the tip vortices increases so slowly that an effective reduction of the angle of attack does not occur until the lift has risen nearly to its asymptote.

Lift with Elliptical Span Loading

The foregoing calculations of $\alpha_e(s)$ and $\Gamma(s)$ were applied to wings with finite tip vortices. Any preassigned variation of the loading across the span may be built up, however, by the addition of elementary vortex loops.

The expression for the downwash at the center of the wing may be integrated in the case of elliptic loading. Let

$$\gamma = \gamma_o \sin \theta \tag{14}$$

$$d\gamma = \gamma_o \cos \theta \, d\theta \tag{15}$$

and let $y = \frac{b}{2} \cos \theta$

$$y^2 + x^2 = \left[\left(\frac{b}{2}\right)^2 + x^2\right]\left[1 - k^2 \sin^2 \theta\right] \tag{16}$$

where

$$k = \sqrt{\frac{\left(\frac{b}{2}\right)^2}{\left(\frac{b}{2}\right)^2 + x^2}}$$

Then the downwash due to a series of loops of the form CEF (fig. 4) is

$$w = \frac{1}{2\pi}\gamma_o \int_0^{\pi/2} \left(\frac{x}{\frac{b}{2}\sqrt{x^2+\left(\frac{b}{2}\right)^2}} + \frac{\frac{b}{2}\cos^2\theta}{x\sqrt{x^2+\left(\frac{b}{2}\right)^2}}\right) \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}} \tag{17}$$

$$w = \frac{1}{2\pi}\gamma_o \left\{ \frac{xk}{\left(\frac{b}{2}\right)^2} K(k) + \frac{1}{x}\left[K(k)\left(k - \frac{1}{k}\right) + \frac{1}{k}E(k)\right]\right\} \tag{18}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals. (See Peirce's table.)

The induced downwash per unit circulation around the wing, as defined under The Downwash Correction, is

$$\frac{dw_i}{d\gamma}(x) = \frac{1}{2\pi}\left\{ \frac{xk}{\left(\frac{b}{2}\right)^2} K(k) + \frac{1}{x}\left[K(k)\left(k - \frac{1}{k}\right) + \frac{1}{k}E(k) - 1\right]\right\} \tag{19}$$

It is necessary, as before, to find the equivalent length x from the distance s. (See equation (8).)

The indicial-lift functions (fig. 8) were determined by graphical integrations, as in the preceding case.

OPERATIONAL FORMULAS FOR THE LIFT

Indicial Lift Functions

The lift of a wing under various dynamical conditions may be conveniently found by the operational method described in reference 5. In order to facilitate the use of the lift functions in such problems, approximate formulas, together with their operational equivalents, have been devised:

$$C_{L_1}(s) = C_0 + C_1 e^{r_1 s} + C_2 e^{r_2 s} \tag{20}$$

Values of the constants are listed in the following table:

TABLE I

A	C_0	C_1	r_1	C_2	r_2
∞	2π	-0.330π	-0.0455	-0.670π	-0.300
6	4.71	-1.740	-.324	0	0
3	3.77	-1.07	-.490	0	0

Points shown in figure 8 illustrate the degree of exactness of these formulas.

The operational equivalents of the preceding expressions are readily formed from the relation

$$e^{rs} = \frac{D}{D - r} 1(s) \tag{21}$$

whence

$$C_{L_1}(s) = \bar{C}_L(D)\ 1(s) = \left(C_0 + C_1 \frac{D}{D - r_1} + C_2 \frac{D}{D - r_2}\right) 1(s) \tag{22}$$

Lift Functions for an Oscillating Airfoil

Following the general formula

$$C_L(s) = \bar{C}_{L_1}(D)\ \alpha(s) \tag{23}$$

the lift in sinusoidal motion, where

$$\alpha = e^{ins} \tag{24}$$

is given by

$$C_{L_n}(s) = \bar{C}_{L_1}(D)e^{ins} = \bar{C}_{L_1}(D)\ \frac{D}{D - in}\ 1(s) \tag{25}$$

where the instantaneous lift due to acceleration is omitted. On expansion of this operator it is found that, with the exception of transient terms,

$$C_{L_n}(s) = \bar{C}_{L_1}(in)\ e^{ins} \tag{26}$$

The real and the imaginary parts of $\bar{C}_{L_1}(in)$ correspond to the functions $F(n)$ and $G(n)$ developed for the two-dimensional case by Theodorsen (reference 2):

$$\bar{C}_{L_1}(in) = 2\pi\ [F(n) + iG(n)] \tag{27}$$

where

$$2\pi F = C_0 + C_1\ \frac{n^2}{{r_1}^2 + n^2} + C_2\ \frac{n^2}{{r_2}^2 + n^2}$$

$$2\pi G = -C_1\ \frac{r_1 n}{{r_1}^2 + n^2} - C_2\ \frac{r_2 n}{{r_2}^2 + n^2}$$

Figure 9 shows these functions as calculated from the values listed in table I.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., December 14, 1938.

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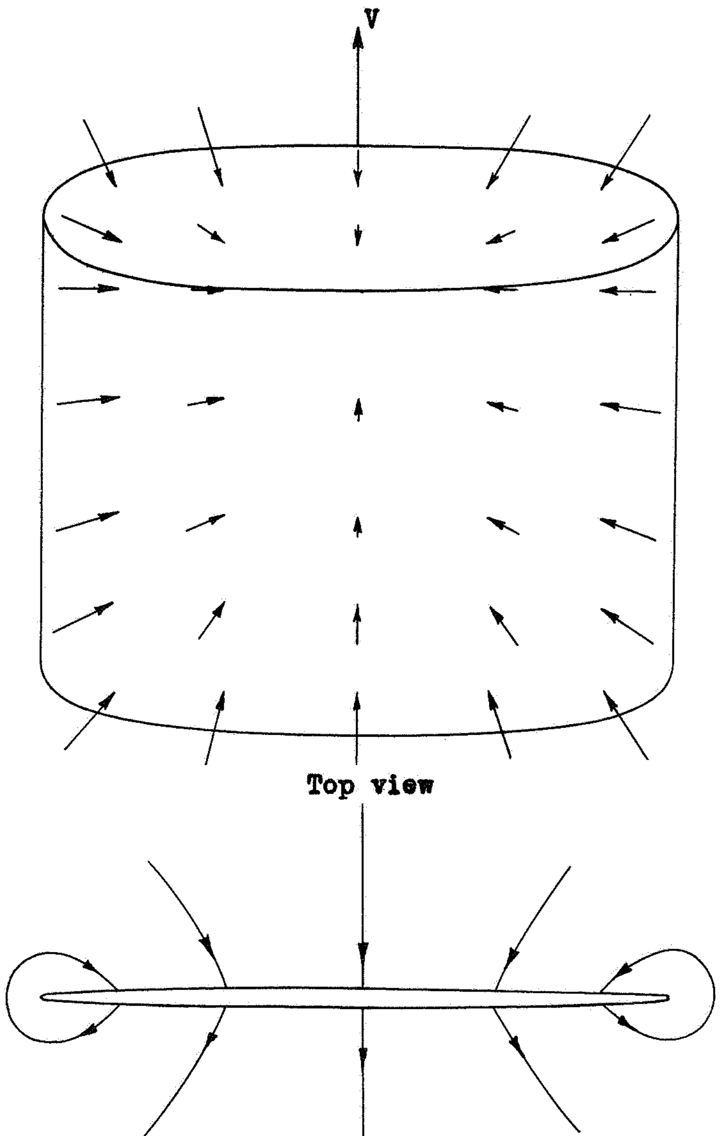

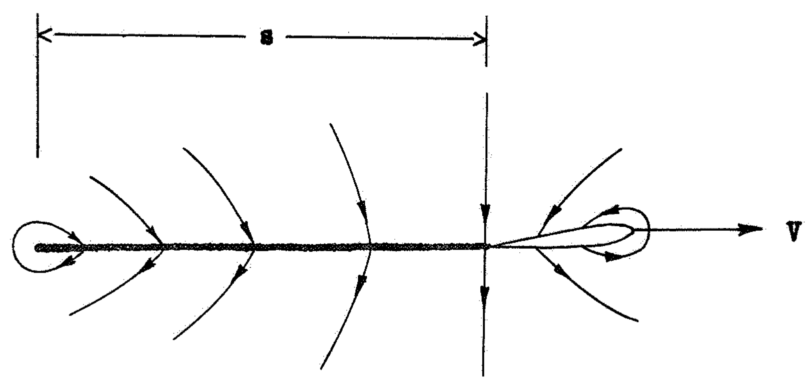

Figure 1.- Flow caused by finite wing, showing surface of discontinuity.

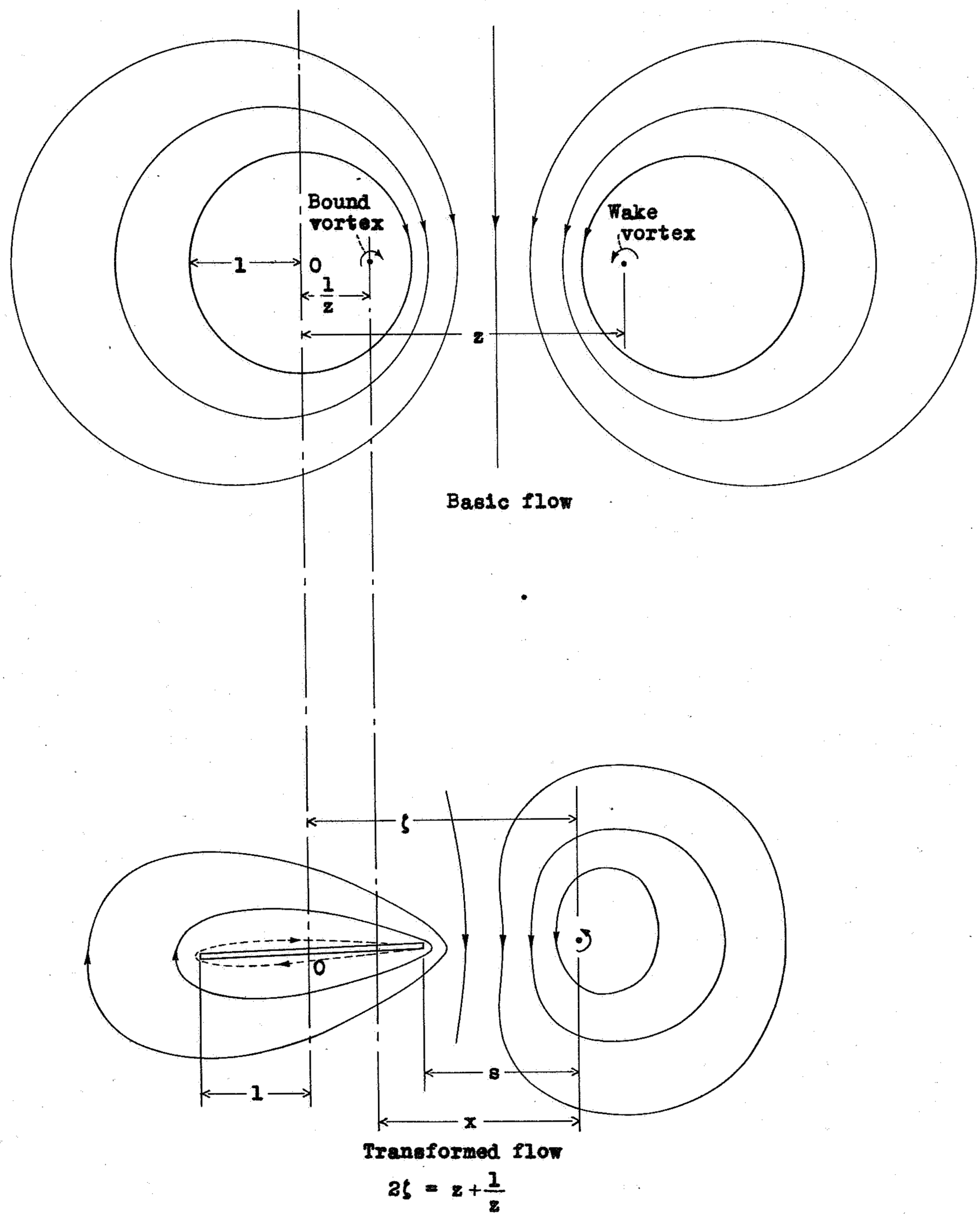

Figure 2.- Element of circulatory flow around wing section.

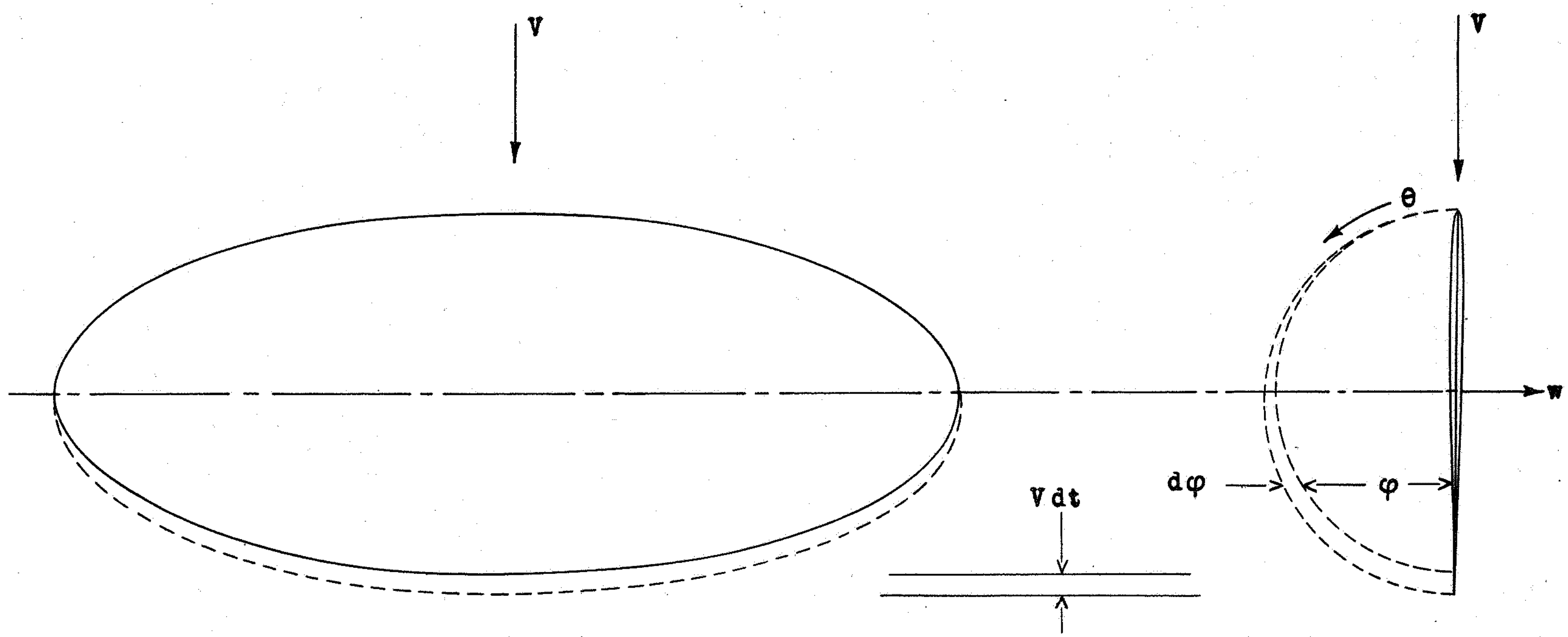

Figure 3.- The wake and the distribution of potential over the chord shortly after the start.

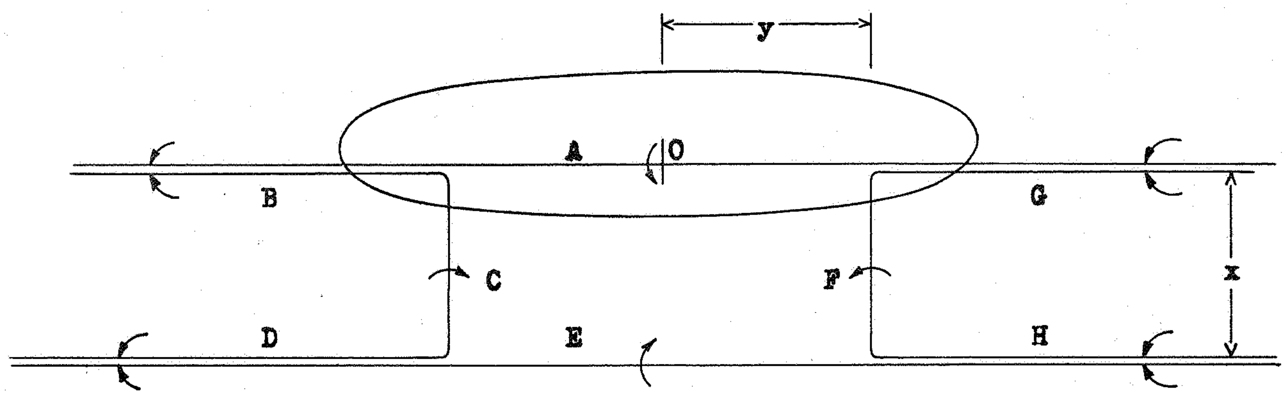

Figure 4.- Superposition of vortices to obtain finite loop.

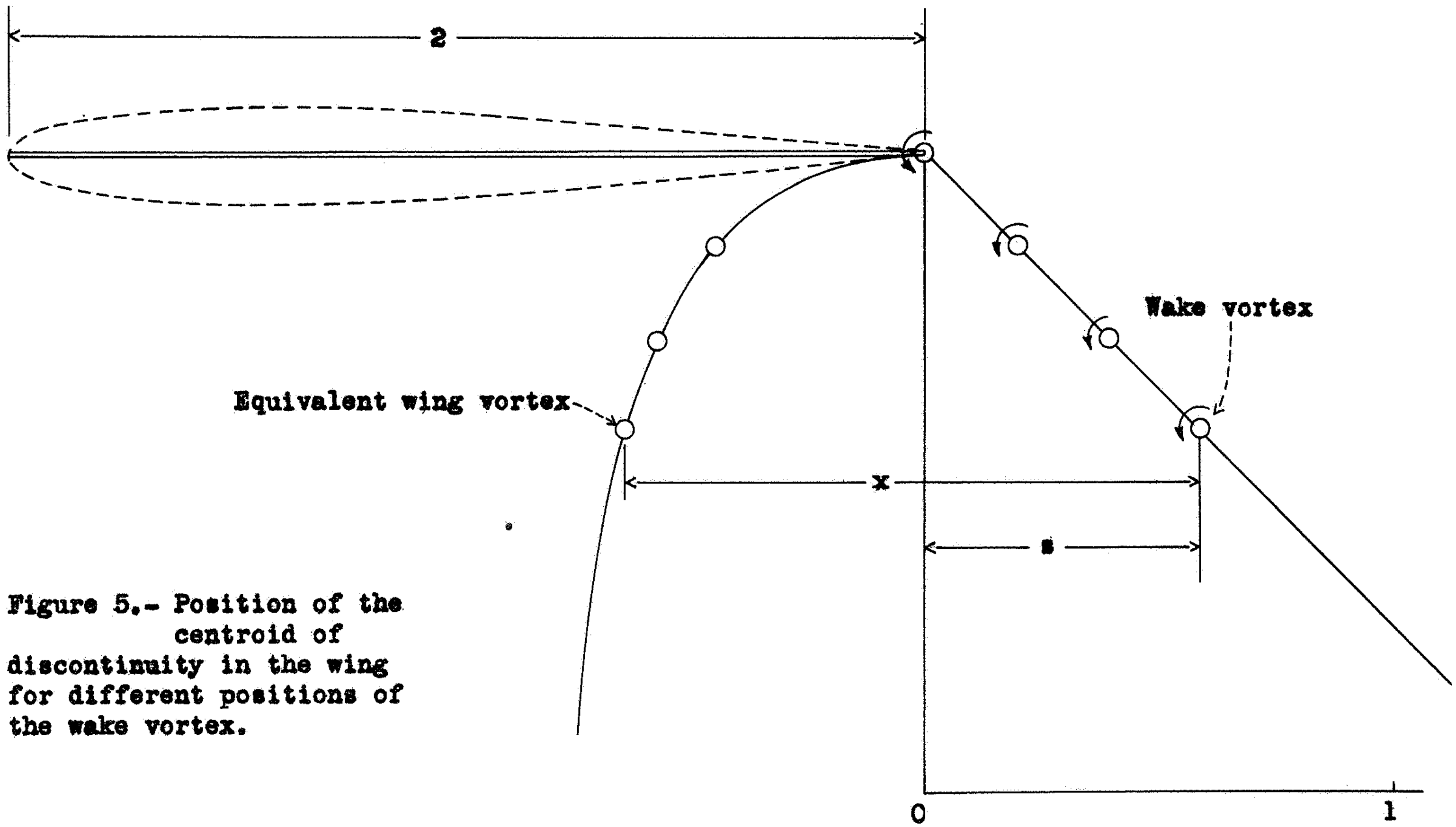

Figure 5.- Position of the centroid of discontinuity in the wing for different positions of the wake vortex.

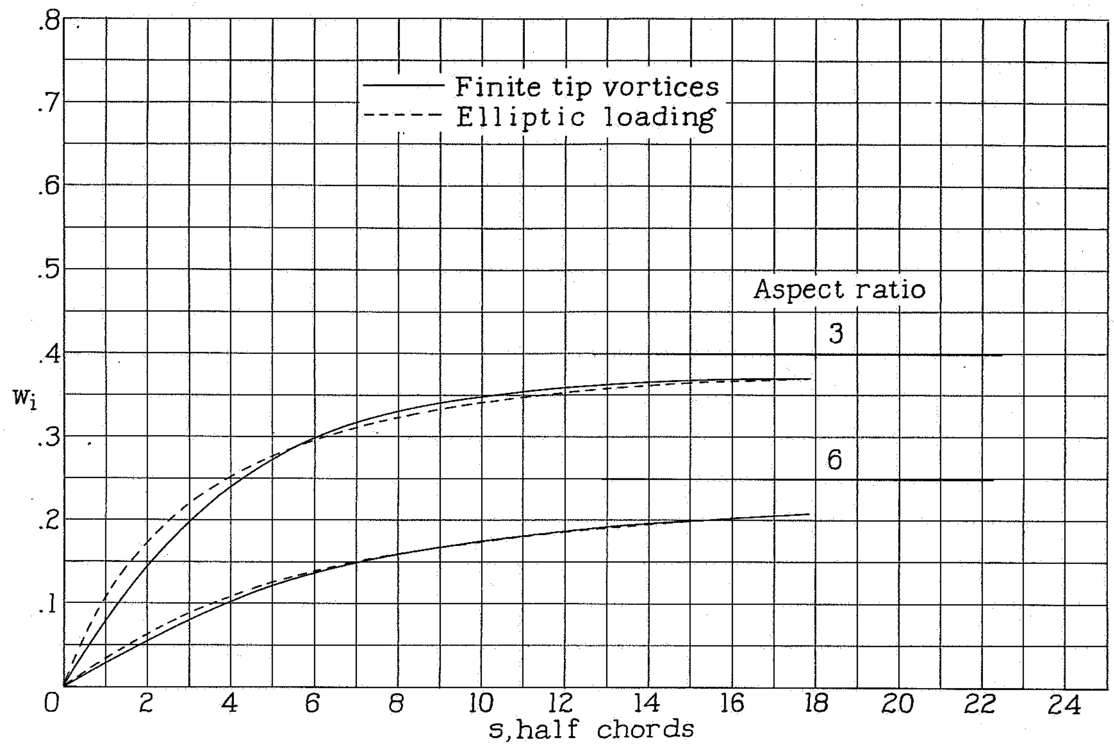

Figure 6.- Indicial-downwash curves.

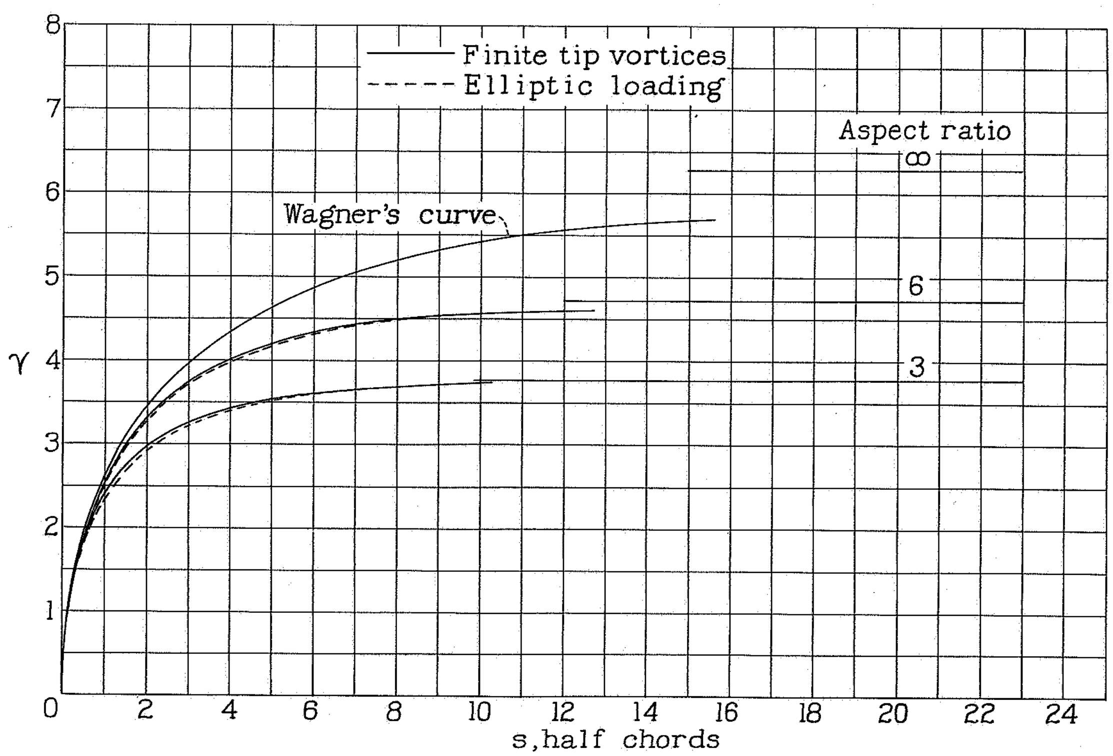

Figure 7.- Indicial-circulation curves.

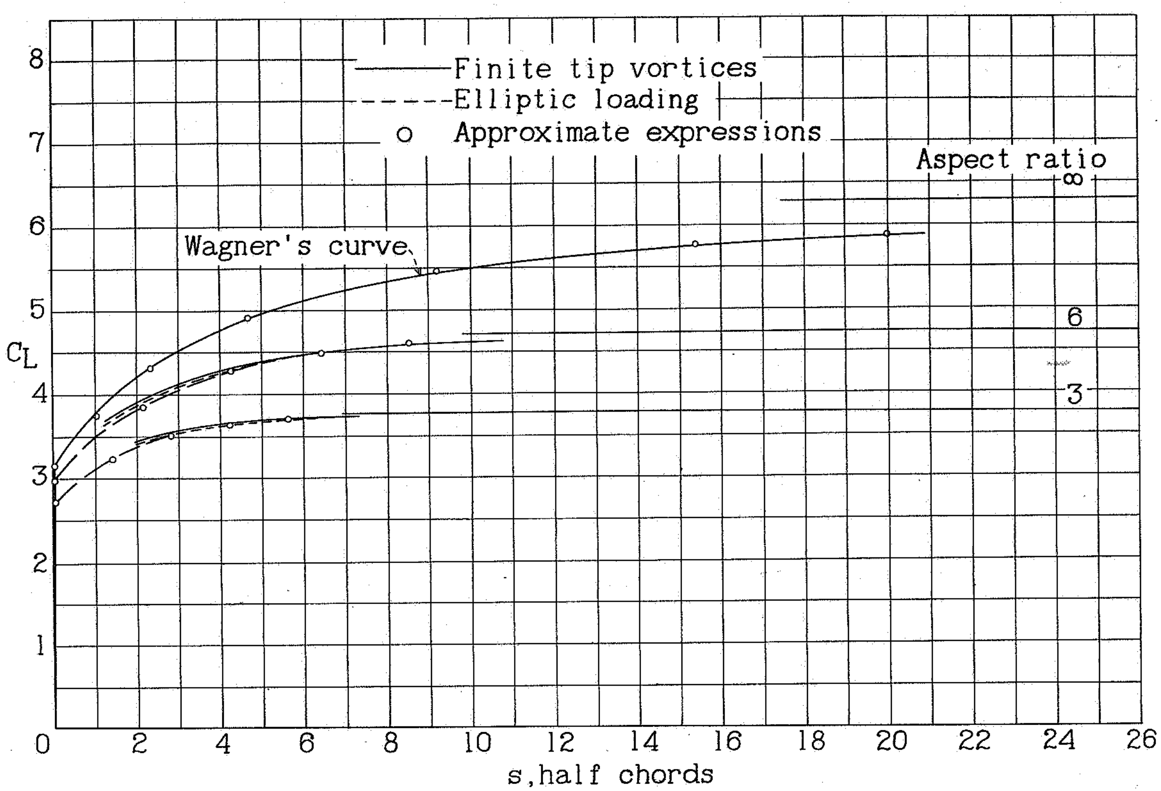

Figure 8.- Indicial-lift functions.

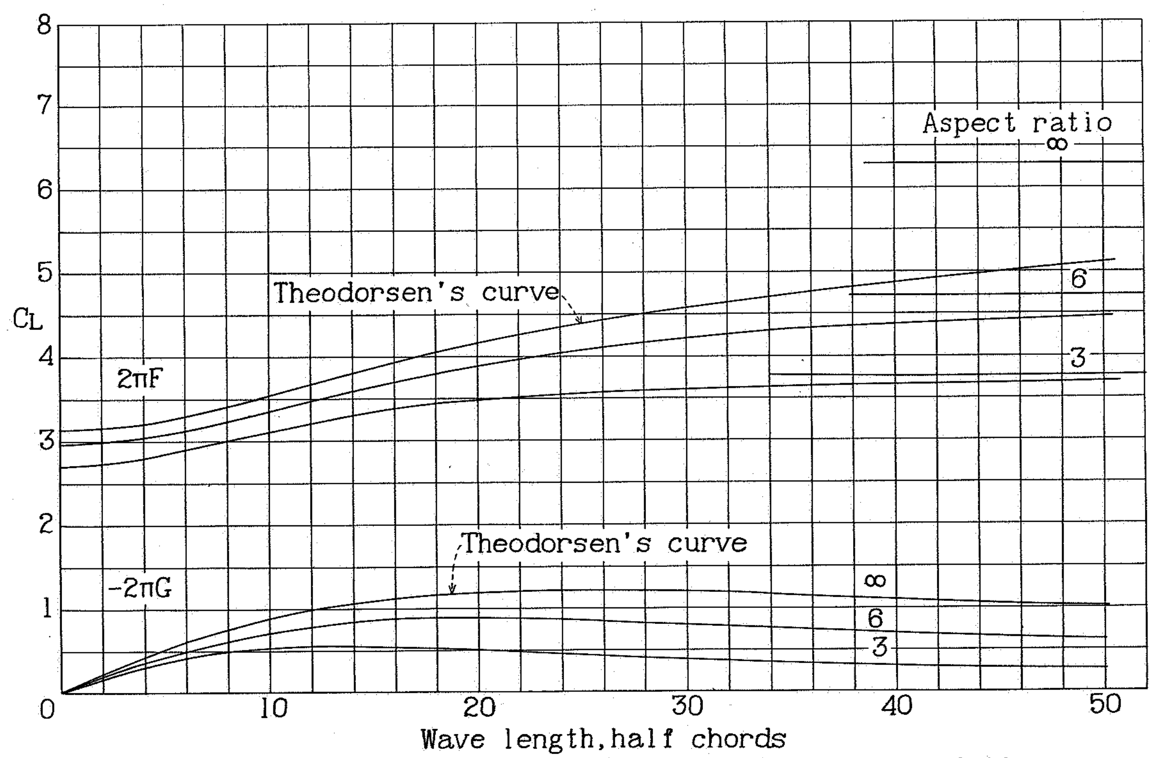

Figure 9.- Lift functions for an oscillating airfoil.